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複体に関する EDWARDS-WALSH RESOLUTIONS と ABELIAN GROUPS

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1. INTRODUCTION

The purpose of this note is to introduce my recent work [15] about cohomological dimension and resolutions of complexes. We recall that the covering dimension $\dim X$ of a compactum X is the smallest natural number n such that there exists an $(n+1)$-fold covering by arbitrarily fine open sets. The characterization of dimension in terms of mappings to spheres led to the cohomological characterization of dimension under the assumption of finite-dimensionality of a space [8]. This characterization was the point of departure for cohomological dimension theory. We give below the definition of cohomological dimension. The cohomological dimension $\text{c-dim}_G X$ of a compactum X with coefficients in an abelian group G is the largest integer n such that there exists a closed subset A of X with $H^n(X, A; G) \neq 0$, where $H^n(\ \ ; G)$ means the Čech cohomology with coefficients in G. Clearly, $\dim X \leq n$ implies that $\text{c-dim}_G X \leq n$ for all G. Alexandroff formulated the theory in his paper [1].

Recent progress of cohomological dimension theory follows from R.D.Edwards theorem [6] (details can be found in [13]). The theorem is based on the excellent idea, which is the so-called *Edwards-Walsh modification*. An equivalent reformulation below caused the advances: associating to each simplicial complex L, a combinatorial resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ (see Definition 2.1 below) specified that $\text{c-dim}_G X \leq n$ if and only if for every simplicial complex L and map $f\colon X \to L$, there exists an approximate lift $\tilde{f}\colon X \to \mathrm{EW}_G(L,n)$ of f; see [5]. Recent analyses in the theory led to a need for those resolutions for general groups. By reason of the necessity, Dydak-Walsh [5, Theorem 3.1] stated a necessary and sufficient condition for the existence of an Edwards-Walsh resolution of an $(n+1)$-dimensional simplicial complex. They [5, Theorem 4.1] also analyzed the modification and investigated a general property of an abelian group G that admits such a resolution of a complex.

For reason of a difficulty, Koyama and the author [11] introduced a property of an abelian group G that induces the existence of an Edwards-Walsh resolution of a simplicial complex: an abelian group G has *property* (EW) provided that there exists a homomorphism $\alpha\colon \mathbf{Z} \to G$ such that

(EW_1) $\alpha \otimes \mathrm{id}\colon \mathbf{Z} \otimes G \to G \otimes G$ is an isomorphism, and
(EW_2) $\alpha^*\colon \mathrm{Hom}(G,G) \to \mathrm{Hom}(\mathbf{Z},G)$ is an isomorphism.

Throughout this note, $\mathbf{Z}$ is the additive group of all integers and $\mathbf{Q}$ is the additive group of all rational numbers. $\mathbf{Z}_{(P)}$ is the ring of integers localized at a subset P of

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$\mathcal{P} = \{\text{all prime numbers}\}$. We denote by $\mathbf{Z}/p$, $\mathbf{Z}/p^\infty$ and $\hat{\mathbf{Z}}_p$ the cyclic group of order p, the quasi-cyclic group of type p^∞ and the group of p-adic integers, respectively.

For a brief historical view of cohomological dimension theory, we refer the reader to [2], [4], [9] and [10].

2. Edwards-Walsh resolutions of complexes

As mentioned above, an important tool of characterizing compacta X with finite cohomological dimension with respect to G is an Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ of a simplicial complex L. For $G = \mathbf{Z}$, those resolutions were formulated in [13]. The relation of Edwards-Walsh resolutions to cohomological dimension theory and their existence for certain other groups were discussed in [3] and [5].

Definition 2.1. Let G be an abelian group and L a simplicial complex. An *Edwards-Walsh resolution* of L in the dimension n is a pair $(\mathrm{EW}_G(L,n),\omega)$ consisting of a CW-complex $\mathrm{EW}_G(L,n)$ and a combinatorial map $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ (that is, $\omega^{-1}(|L'|)$ is a subcomplex for each subcomplex L' of L) such that

(i) $\omega^{-1}(|L^{(n)}|) = |L^{(n)}|$ and $\omega|_{|L^{(n)}|}$ is the identity map of $|L^{(n)}|$ onto itself,
(ii) for every simplex σ of L with $\dim\sigma > n$, the preimage $\omega^{-1}(\sigma)$ is an Eilenberg-MacLane complex of type $(\bigoplus G, n)$, where the sum here is finite, and
(iii) for every simplex σ of L with $\dim\sigma > n$, the inclusion $\omega^{-1}(\partial\sigma) \to \omega^{-1}(\sigma)$ induces an epimorphism $H^n(\omega^{-1}(\sigma);G) \to H^n(\omega^{-1}(\partial\sigma);G)$.

Dydak-Walsh established a property of G that characterizes those groups for which such resolutions exist for all $(n+1)$-dimensional simplicial complexes.

Theorem [5, Theorem 3.1]. *Let G be an abelian group and $n \geq 1$. An Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ exists for all simplicial complexes L with $\dim L \leq n+1$ if and only if there exists an integer $m \geq 1$ and a homomorphism $\alpha\colon \mathbf{Z} \to G^m$ such that any homomorphism $\beta\colon \mathbf{Z} \to G$ factors as $\beta = \tilde{\beta}\circ\alpha$ for some $\tilde{\beta}\colon G^m \to G$.*

We extend the theorem above to all simplicial complexes of dimension $\geq n+2$. Before stating our theorem, we recall a proposition in [11].

Proposition 2.2. *Let σ be an $(n+2)$-simplex and $(K(G,n),S^n)$ a pair of an Eilenberg-MacLane complex of type (G,n) and an n-dimensional sphere S^n in $K(G,n)$. Let E be the CW-complex obtained by replacing each $(n+1)$-face τ of $\partial\sigma$ by $(K(G,n),S^n)$ along $\partial\tau \cong S^n$. Then we have*

$$H_n(E) \approx (G/\operatorname{Im}\alpha) \oplus \underbrace{G\oplus\cdots\oplus G}_{n+2}$$

and an exact sequence

$$\mathbf{Z} \xrightarrow{\Delta_\alpha} \underbrace{G\oplus\cdots\oplus G}_{n+3} \xrightarrow{q} (G/\operatorname{Im}\alpha) \oplus \underbrace{G\oplus\cdots\oplus G}_{n+2} \longrightarrow 0,$$

where $\alpha = \pi_n(S^n \hookrightarrow K(G,n))$ and Δ_α and q are given by

$$\Delta_\alpha(j) = (\alpha(j), -\alpha(j), \dots, -\alpha(j))$$

and

$$q((g_0, g_1, \dots, g_{n+2})) = ([g_0], g_1 + g_0, \dots, g_{n+2} + g_0).$$

The next is our main theorem.

Theorem 2.3. *Let $\alpha\colon \mathbf{Z} \to G$ be a homomorphism from the group of integers to an abelian group G. Then the following are equivalent:*

(1) *there exists an Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L, n) \to |L|$ of each simplicial complex L with $\dim L \geq n + 2$ such that*
  (iv) *the inclusion-induced homomorphism $\pi_n(\omega^{-1}(\partial\tau)) \to \pi_n(\omega^{-1}(\tau))$ is α for each $(n + 1)$-simplex τ of L, and*
  (v) *the inclusion-induced homomorphism $\pi_n(\omega^{-1}(\partial\sigma)) \to \pi_n(\omega^{-1}(\sigma))$ maps the subgroup $G/\operatorname{Im}\alpha$ to zero for any $(n + 2)$-simplex σ of L (where if $n = 1$, we consider the abelianization of the fundamental groups),*

(2) *the homomorphism $\alpha^*\colon \mathrm{Hom}(G, G) \to \mathrm{Hom}(\mathbf{Z}, G)$ induced by α is an isomorphism.*

Remark 2.4. The subgroup $G/\operatorname{Im}\alpha$ in condition (v) above depends upon the enumeration of $(n+1)$-faces of each $(n+2)$-simplex, since we calculate the group by Proposition 2.2. We also note that (v) is natural for constructing our desired resolution.

Remark. The groups $\mathbf{Z}$, $\mathbf{Z}/p$ and $\mathbf{Z}_{(p)}$ satisfy such a condition, that is, there are such resolutions with respect to the groups (those are well-known [13], [5] and [2, 3]).

Example. If $G = \mathbf{Z}/p \oplus \mathbf{Z}_{(q)}$ or $\hat{\mathbf{Z}}_p$, where $p \neq q$, then Edwards-Walsh resolutions $\omega\colon \mathrm{EW}_G(L, n) \to |L|$ exist for all n and all simplicial complexes.

As we have previously stated, property (EW) seems strong to construct a resolution. However, the condition group-theoretically give us an interesting future.

Theorem 2.5. *Let G be an abelian group with property* (EW). *Then the group is precisely either a cyclic group or a localization of the integer group at some prime numbers.*

Remark. We note that if G is either a cyclic group or a localization of the integer group at some prime numbers, then G has property (EW). Thus the condition characterizes the group of integers and the Bockstein groups except quasi-cyclic ones.

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